Projective and spherical trigonometry

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Introduction

Spherical trigonometry is historically one of the most important areas of mathematics, due to the obvious applications to astronomy and navigation. *Planar trigonometry*, on the other hand, was for fifteen centuries only of marginal interest, and the most important formulas were developed first in the spherical case, even though they are more difficult there. Only in the last four hundred years or so, with the advent of modern technology and engineering, did planar trigonometry slowly begin to gain the ascendancy. This reversal is now complete, and the modern curriculum largely ignores spherical geometry. Perhaps the many complicated formulas present too much of a hurdle.

Recently a new framework for planar trigonometry has been proposed ([Wildberger]). *Rational trigonometry* replaces distance and angle with quadratic concepts called *quadrance* and *spread*. The usual laws are replaced by purely algebraic analogs, with the consequence that they hold in much wider generality, allow more accurate calculations, and are much easier to learn. The usual menagerie of transcendental circular functions and their inverses play no role.

Astronomy, however, requires understanding of how the earth's rotation affects our view of the sky and the objects in it. As a consequence, longitudinal angle becomes an important and unavoidable concept. But for many spherical geometrical applications, there is no uniform motion around a fixed axis that plays such a distinguished role. For this kind of 'stationary' spherical geometry, it turns out that there is a rational version of the classical theory which again is simpler, more elegant and accurate. This theory is here developed in the more natural setting of *projective trigonometry*.

The projective plane inherits a rich metrical structure which extends to higher dimensions and arbitrary fields, a fact which has major implications for algebraic geometry, and possibly also for differential geometry. Thales' theorem and Pythagoras' theorem are particularly important, and the wide variety of classical spherical formulas are replaced by simpler, polynomial relations. The Platonic solids are seen in a new light. In a future article I will show that the formulas given here hold in hyperbolic geometry too—essentially without any modification.

Rational trigonometry

In this preliminary section we present some basic facts about (Euclidean) rational trigonometry, taken from [Wildberger] and suitably modified to hold in three dimensional space.

The **quadrance** $Q(A_1, A_2)$ between the points $A_1 \equiv [x_1, y_1, z_1]$ and $A_2 \equiv [x_2, y_2, z_2]$ is the number

$$Q(A_1, A_2) \equiv (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 .$$

Quadrance is just distance squared over the decimal numbers, although the algebraic definition extends to arbitrary fields. The points A_1, A_2 and A_3 are collinear precisely when the quadrances

$$Q_1 \equiv Q(A_2, A_3) \qquad Q_2 \equiv Q(A_1, A_3) \qquad Q_3 \equiv Q(A_1, A_2)$$

satisfy the **Triple quad formula**

$$(Q_1 + Q_2 + Q_3)^2 = 2\left(Q_1^2 + Q_2^2 + Q_3^2\right). \tag{1}$$

The points A_1, A_2 and A_3 form a right triangle with the lines A_1A_3 and A_2A_3 perpendicular precisely when the quadrances satisfy **Pythagoras' theorem**

$$Q_1 + Q_2 = Q_3.$$

The notion of angle between two lines l_1 and l_2 is replaced by that of the spread $s(l_1, l_2)$ which is a (dimensionless) number between 0 and 1, and which is unchanged if either l_1 or l_2 is translated. To define it, suppose that l_1 and l_2 intersect at the point A. Choose a point $B \neq A$ on one of the lines, say l_1, and let C be the foot of the perpendicular from B to l_2 as in Figure 1. If $Q(B, C) = Q$ and $Q(A, B) = R$ then define

$$s = s(l_1, l_2) = \frac{Q}{R}.$$

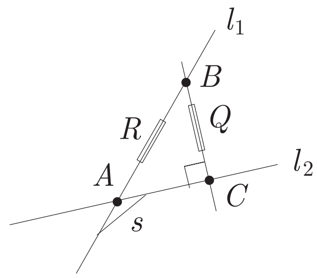

Figure 1: Spread as ratio

The spread between parallel lines is 0. The spread corresponding to 30° or 150° is $s = 1/4$, to 45° or 135° is 1/2, and to 60° or 120° is 3/4. The spread between perpendicular lines is 1. It is not hard to check that if $O = [0, 0, 0]$, $A_1 \equiv [x_1, y_1, z_1]$ and $A_2 \equiv [x_2, y_2, z_2]$ then the spread $s = s(OA_1, OA_2)$ is

$$s = \frac{(y_1z_2 - z_1y_2)^2 + (z_1x_2 - x_1z_2)^2 + (x_1y_2 - y_1x_2)^2}{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)}. \tag{2}$$

The spread may be measured by a spread protractor. The one in Figure 2 was created by Michael Ossmann and may be downloaded at http://www.ossmann.com/protractor/.

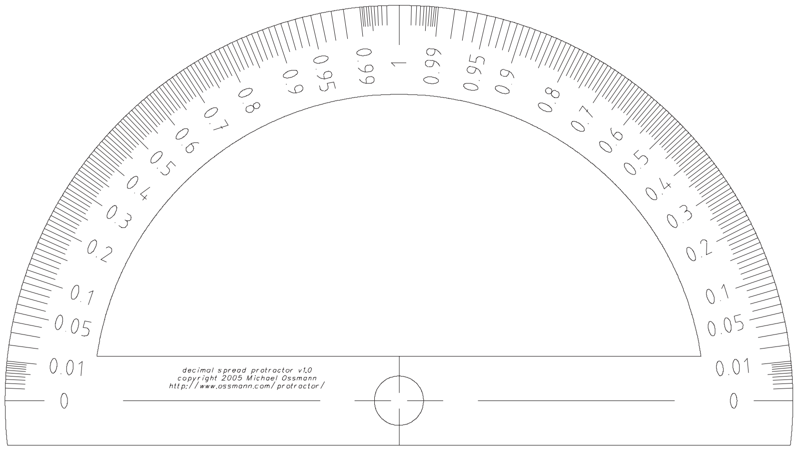

Figure 2: A spread protractor

The notion of spread may be extended to include planes in three dimensions. The **spread** between a line l and a plane Π is defined to be 0 if they are parallel, 1 if they are perpendicular, and otherwise is defined to be the spread between l and the unique line m which is the intersection of Π and the plane perpendicular to Π passing through l.

The **spread** between two planes Π_1 and Π_2 in three dimensional space is defined to be 0 if they are parallel, 1 if they are perpendicular, and otherwise is defined to be the spread between the two lines formed by intersecting Π_1 and Π_2 with a plane Π perpendicular to them both, that is a plane perpendicular to the line of intersection of Π_1 and Π_2.

A **triangle** $\overline{A_1A_2A_3}$ is a set of three non-collinear points. Given three distinct points A_1, A_2 and A_3, define the quadrances Q_1, Q_2 and Q_3 as above and the spreads

$$s_1 \equiv s(A_1A_2, A_1A_3) \qquad s_2 \equiv s(A_2A_1, A_2A_3) \qquad s_3 \equiv s(A_3A_1, A_3A_2).$$

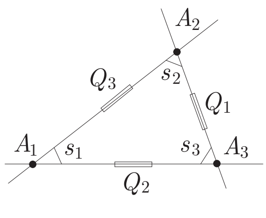

Figure 3: Quadrances and spreads

Here are the other main laws of rational trigonometry.

Spread law

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3}.$$

Cross law

$$\left(Q_1+Q_2-Q_3\right)^2=4Q_1Q_2\left(1-s_3\right).$$

Triple spread formula

$$\left(s_1+s_2+s_3\right)^2=2\left(s_1^2+s_2^2+s_3^2\right)+4s_1s_2s_3.$$

This last law replaces the rule that the sum of a triangle's angles is π, and is a modification of the Triple quad formula. One of its important consequences is that if $l_0,l_1,l_2,l_3,\cdots$ are equally spaced intersecting lines, with common spread $s=s\left(l_0,l_1\right)=s\left(l_1,l_2\right)=\cdots$ then

$$\begin{aligned}
s\left(l_0,l_2\right) &= 4s\left(1-s\right)=S_2\left(s\right)\\
s\left(l_0,l_3\right) &= s\left(3-4s\right)^2=S_3\left(s\right)\\
s\left(l_0,l_4\right) &= 16s\left(1-s\right)\left(1-2s\right)^2=S_4\left(s\right)\\
s\left(l_0,l_5\right) &= s\left(5-20s+16s^2\right)^2=S_5\left(s\right)
\end{aligned}$$

and so on. The first of these statements is called the **Equal spreads theorem** and is very useful. Note that $S_2\left(s\right)$ is the logistic map of chaos theory fame.

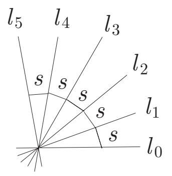

Figure 4: Multiple spreads

The fifth spread polynomial $S_5\left(s\right)=s\left(5-20s+16s^2\right)^2$ controls five-fold symmetry, and over the decimal field has non-trivial zeroes

$$\alpha=\left(5-\sqrt{5}\right)/8\approx 0.345491503\ldots \qquad \text{and} \qquad \beta=\left(5+\sqrt{5}\right)/8\approx 0.904508497\ldots$$

which are spreads in the regular pentagon of Figure 5. Note that $S_2\left(\alpha\right)=\beta$ and $S_2\left(\beta\right)=\alpha$.

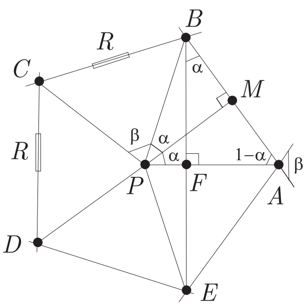

Figure 5: A pentagon

An interesting example

Here is an example that is surprisingly involved considering its simplicity (and has a spherical analog of importance to the Platonic solids). Take an equilateral triangle $\overline{ABC}$ and draw lines AD, BE and CF as in Figure 6, all making the same spreads with the sides of the triangle, so that $\overline{DEF}$ is equilateral, and so that

$$Q(A,D) = Q(D,E) = Q(B,E) = Q(E,F) = Q(C,F) = Q(F,D).$$

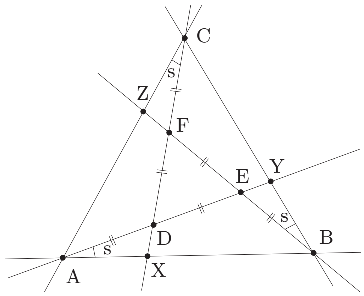

Figure 6: An equilateral arrangement

The spread $s = s(AC, CD)$ turns out to be $3/28$, and the corresponding angle is about 19.11°. If we complete the configuration to make it more symmetrical, then numerous other spreads and quadrances may be calculated using the basic laws above, resulting in Figure 7. The quadrances have been normalized so they are all commensurable. Note that the segment $\overline{AX}$ trisects $\overline{AB}$, and that $Q(A,X) : Q(D,X) = 7 : 1$. Checking the numbers in this Figure is an excellent exercise.

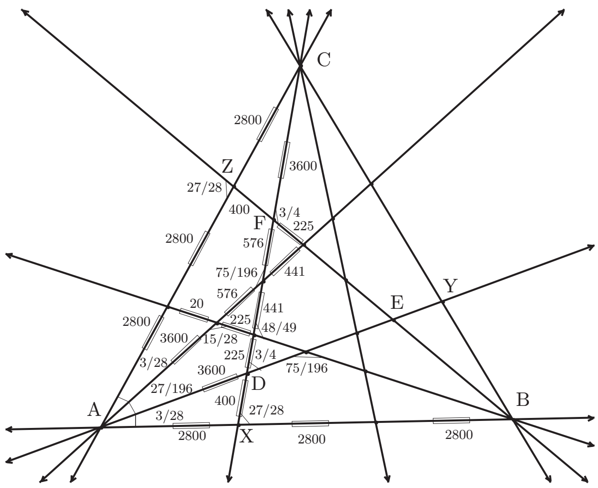

Figure 7: More quadrances and spreads

Rational projective trigonometry

The sphere has equation $x^2 + y^2 + z^2 = 1$ and center $O = [0, 0, 0]$. Any two non-antipodal points A and B lying on it determine a unique spherical line, or great circle arc, which is the intersection of the sphere with the plane OAB. Any two such spherical lines intersect at a pair of antipodal points.

The existence of antipodal points is somewhat awkward, so nineteenth century geometers considered also *elliptic geometry*, the result of identifying antipodal points on the sphere. An alternative, somewhat superior, formulation is to consider the associated line through the origin O passing through the antipodal points. Such a line will be called a **projective point**. Similarly a plane through O will be called a **projective line**. Figure 8 shows on the left a spherical triangle formed by three spherical points A, B and C and three great circle arcs, and on the right the corresponding projective triangle, consisting of three projective points a, b and c, and the three projective lines that they form.

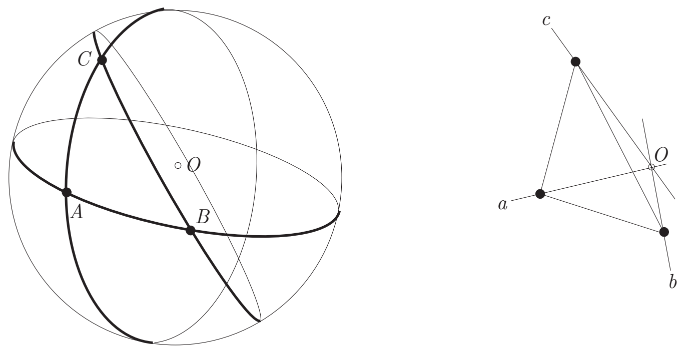

Figure 8: Spherical and projective triangles

There are several good reasons to view the projective triangle as the primary object. The ambiguity afforded by antipodal points on the sphere disappears, the projective space is more general over arbitrary fields (the 2 : 1 correspondence between spherical points and projective points is a property of the decimal number field), and the projective space is the natural domain of algebraic geometry. But the best reason is that in the projective plane there is a complete *duality* between projective points a and projective lines L, as every plane L through the origin has a unique normal line a through the origin, and conversely. We now establish *rational projective trigonometry,* but illustrate and apply the theory also to spherical triangles.

The projective point a **lies on** the projective line L, or equivalently L **passes through** a, precisely when the line a lies on the plane L. Two distinct projective points a_1 and a_2 determine a unique projective line $L = a_1 a_2$ which passes through them both. Two distinct projective lines L_1 and L_2 determine a unique projective point $a = L_1 L_2$ which lies on them both.

If $O = [0, 0, 0]$ and $A = [x, y, z]$ then the projective point $a = OA$ will be written $a = [x : y : z]$. The projective line with equation $lx + my + nz = 0$ will be written $L = \langle l : m : n \rangle$. Finding the projective line passing through two projective points (or dually the projective point lying on two projective lines) is essentially the computation of a cross product, so that for example

$$[1 : 2 : 4]\,[3 : 7 : 5] = \langle 2 \times 5 - 4 \times 7 : 4 \times 3 - 1 \times 5 : 1 \times 7 - 2 \times 3 \rangle = \langle -18 : 7 : 1 \rangle .$$

Notice that this product is *commutative*. Two projective points a_1 and a_2 are **perpendicular** precisely when they are perpendicular as lines. Similarly two projective lines are **perpendicular** precisely when they are perpendicular as planes. Given a projective point a and a projective line L, there is always a projective line N which passes through a and is perpendicular to L. This projective altitude is unique unless a is perpendicular to L, in which case any projective line through a has the required property.

Projective points a_1, a_2 and a_3 are **collinear** precisely when they all lie on a single projective line L, and projective lines L_1, L_2 and L_3 are **concurrent** precisely when they all pass through a single projective point a. A **projective triangle** $\overline{a_1a_2a_3}$ is a set of three non-collinear projective points. It determines the three non-concurrent projective lines $L_1 = a_2a_3$, $L_2 = a_3a_1$ and $L_3 = a_1a_2$, called the **projective lines** of the projective triangle.

The **projective quadrance** $q(a_1, a_2)$ between two projective points a_1 and a_2 is defined to be the spread between them. After all, a_1 and a_2 are two lines intersecting at O, and so they have a well-defined spread. If $a_1 = [x_1 : y_1 : z_1]$ and $a_2 = [x_2 : y_2 : z_2]$ then $q(a_1, a_2)$ is exactly the expression (2).

The **projective spread** $S(L_1, L_2)$ between two projective lines L_1 and L_2 is defined to be the spread between them. After all, L_1 and L_2 are two planes intersecting at O, and so they have a well defined spread. In terms of coordinates, this projective spread has the same form as (2). Duality ensures that all formulas in the subject have a dual formulation, where the projective points and projective lines interchange, and the corresponding projective quadrances and projective spreads interchange.

Given three distinct projective points a_1, a_2 and a_3, and the associated projective lines $L_1 = a_2a_3$, $L_2 = a_1a_3$ and $L_3 = a_1a_2$, define the projective quadrances

$$q_1 = q(a_2, a_3) \qquad q_2 = q(a_1, a_3) \qquad q_3 = q(a_1, a_2)$$

and the projective spreads

$$S_1 = S(L_2, L_3) \qquad S_2 = S(L_1, L_3) \qquad S_3 = S(L_1, L_2).$$

Such a situation will be designated as follows in Figure 9, either as a spherical figure on the left, a projective figure in the middle, or as a more two dimensional representation of it on the right. Note the opposite use of small and large case letters from rational (Euclidean) trigonometry.

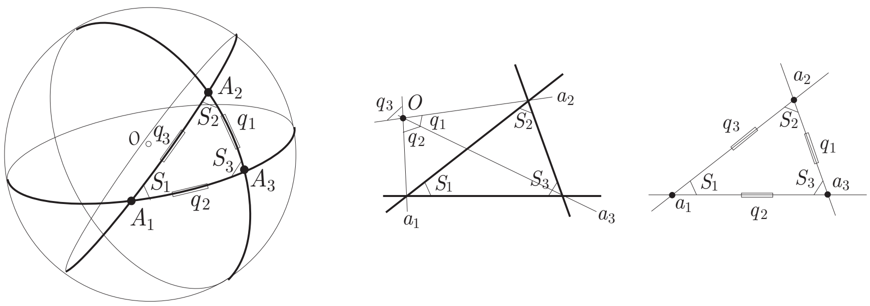

Figure 9: Three views of a projective triangle

Main laws of projective trigonometry

The main laws of projective trigonometry are the Projective versions of Thales' theorem, Pythagoras' theorem, the Triple quad formula, the Spread law and the Cross law. Each of these has dual formulations, with the exception of the Projective spread law, which is essentially self-dual. The first result is the spherical or projective analog of arguably the oldest theorem in mathematics, and explains why the *spread* is the crucial ratio in rational trigonometry—the corresponding results do not hold for the rational analogs of cos and tan.

Theorem 1 (Projective Thales' theorem) *Suppose L_1 and L_2 are distinct projective lines intersecting at the projective point a and with a projective spread of S. Choose a projective point $b \neq a$ on one of the lines, say L_1, and let c be the projective point which is the foot of the (or a) perpendicular projective line N from b to L_2 as in Figure 10. If $q(b,c) = q$ and $q(a,b) = r$ then*

$$S = \frac{q}{r}.$$

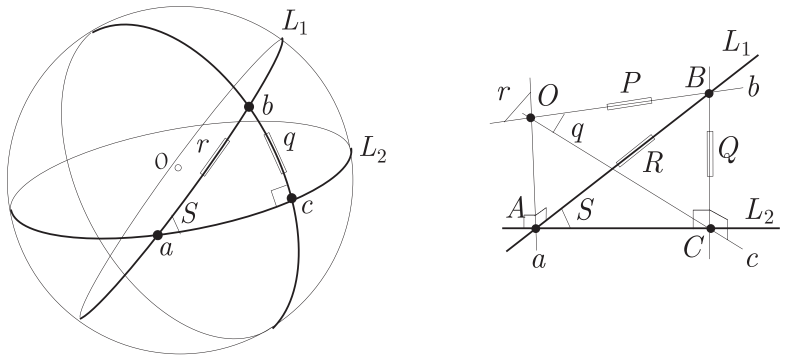

Figure 10: Projective Thales' theorem—spherical and projective views

Proof. If $S = 1$ then L_1 and L_2 are perpendicular. In this case c may be taken to be a, and the equality is true. Otherwise $S \neq 1$ ensures that the perpendicular projective line from b to L_2 is unique, so that c is well-defined. Consider the situation in the projective view of Figure 10.

Choose a point B on b distinct from O. Let C be the foot of the altitude line from B to L_2 so that $OC = c$. Now let A be the foot of the altitude line from C to the line a. Since BC is normal to L_2 it is also perpendicular to a, and so the plane ABC is perpendicular to a. Set $Q \equiv Q(B,C)$, $R \equiv Q(A,B)$ and $P \equiv Q(O,B)$. Then by the definition of the spread S between L_1 and L_2

$$\begin{aligned} S &= s(AB, AF) = \frac{Q}{R} \\ &= \frac{Q/P}{R/P} \\ &= \frac{q}{r}. \end{aligned}$$

■

Here is another diagram to illustrate the situation: the projective spread S can be calculated as 'opposite over hypotenuse projective quadrance' for either of the two right triangles shown in Figure 11.

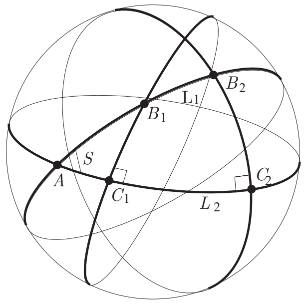

Figure 11: Two ratios give same spread

Theorem 2 (Projective triple quad formula) *If the three projective points* a_1, a_2 *and* a_3 *are collinear, then*

$$(q_1 + q_2 + q_3)^2 = 2\left(q_1^2 + q_2^2 + q_3^2\right) + 4q_1q_2q_3.$$

Proof. This is just a restatement of the Triple spread formula. ■

Theorem 3 (Dual projective triple quad formula) *If the three projective lines* L_1, L_2 *and* L_3 *are concurrent, then*

$$(S_1 + S_2 + S_3)^2 = 2\left(S_1^2 + S_2^2 + S_3^2\right) + 4S_1S_2S_3.$$

Proof. This is dual to the previous result. ■

Theorem 4 (Projective Pythagoras' theorem) *Suppose that* $\overline{a_1a_2a_3}$ *is a projective triangle with projective quadrances* q_1, q_2 *and* q_3*, and projective spreads* S_1, S_2 *and* S_3*. If* $S_3 = 1$ *then*

$$q_3 = q_1 + q_2 - q_1q_2.$$

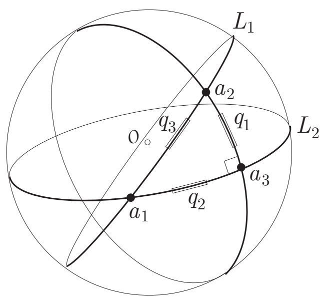

Figure 12: Pythagoras' theorem—spherical view

Proof. Assume the projective triangle $\overline{a_1a_2a_3}$ has projective lines $L_1 = a_2a_3$, $L_2 = a_3a_1$ and $L_3 = a_1a_2$. Suppose that $S_3 = 1$, so that the planes L_1 and L_2 are perpendicular. Let $A_3 \neq O$ be a point lying on a_3, define $H = Q(O, A_3)$ and suppose that the plane Π through A_3 perpendicular to a_3 intersects a_1 and a_2 at A_1 and A_2 respectively.

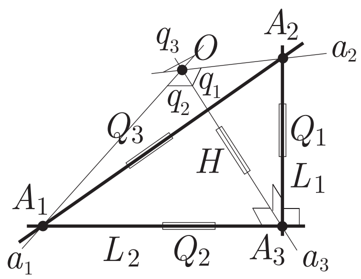

Figure 13: Pythagoras' theorem—projective view

Then by assumption $\overline{A_1A_2A_3}$ is a right triangle, so that its quadrances Q_1, Q_2 and Q_3 satisfy $Q_1 + Q_2 = Q_3$. Since the triangles $\overline{OA_3A_2}$ and $\overline{OA_3A_1}$ are also right triangles,

$$\begin{aligned} Q(O, A_2) &= Q_1 + H \\ Q(O, A_1) &= Q_2 + H. \end{aligned}$$

Thus the Cross law applied to $\overline{OA_1A_2}$ yields

$$((Q_1 + H) + (Q_2 + H) - (Q_1 + Q_2))^2 = 4(Q_1 + H)(Q_2 + H)(1 - q_3).$$

This can be rewritten as

$$1 - q_3 = \left(\frac{H}{Q_1 + H}\right)\left(\frac{H}{Q_2 + H}\right) = (1 - q_1)(1 - q_2).$$

Now rearrange to obtain

$$q_3 = q_1 + q_2 - q_1q_2.$$

■

Theorem 5 (Dual projective Pythagoras' theorem) *Suppose that $\overline{a_1a_2a_3}$ is a projective triangle with projective quadrances q_1, q_2 and q_3, and projective spreads S_1, S_2 and S_3. If $q_3 = 1$ then*

$$S_3 = S_1 + S_2 - S_1S_2.$$

Proof. This is dual to the previous result. ■

Theorem 6 (Projective spread law) *Suppose that $\overline{a_1a_2a_3}$ is a projective triangle with projective quadrances q_1, q_2 and q_3, and projective spreads S_1, S_2 and S_3. Then*

$$\frac{S_1}{q_1} = \frac{S_2}{q_2} = \frac{S_3}{q_3}.$$

Proof. This is analogous to the planar proof in [Wildberger]. Suppose that f is the intersection of a_2a_3 with the perpendicular projective line passing through a_1. As in Figure 14, define the projective quadrances

$$r_1 = q(a_1, f) \qquad r_2 = q(a_2, f) \qquad r_3 = q(a_3, f).$$

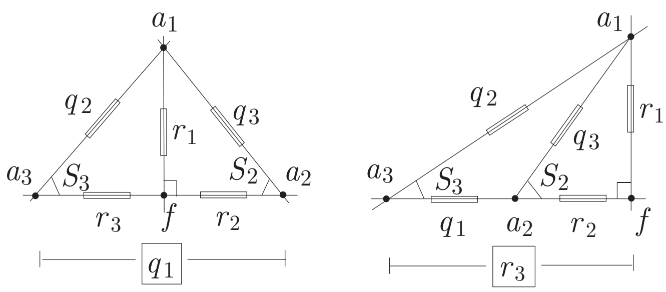

Figure 14: Proofs of the main laws

Apply Projective Thales theorem to see that

$$S_2 = r_1/q_3 \qquad \text{and} \qquad S_3 = r_1/q_2.$$

Solve for r_1 to get

$$r_1 = q_3S_2 = q_2S_3.$$

Thus

$$\frac{S_2}{q_2} = \frac{S_3}{q_3}$$

and similarly

$$\frac{S_1}{q_1} = \frac{S_2}{q_2}.$$

■

Theorem 7 (Projective cross law) *Suppose that $\overline{a_1a_2a_3}$ is a projective triangle with projective quadrances q_1, q_2 and q_3, and projective spreads S_1, S_2 and S_3. Then*

$$(S_3q_1q_2 - q_1 - q_2 - q_3 + 2)^2 = 4(1 - q_1)(1 - q_2)(1 - q_3).$$

Proof. This is also analogous to the planar situation. Use the notation of the previous proof, Projective Thales theorem and Projective Pythagoras' theorem to obtain

$$\begin{aligned} S_3 &= r_1/q_2 \\ q_2 &= r_1 + r_3 - r_1r_3 \\ q_3 &= r_1 + r_2 - r_1r_2. \end{aligned}$$

Now solve sequentially for r_1, r_2 and r_3

$$\begin{aligned} r_1 &= S_3 q_2 \\ r_3 &= \frac{q_2 - S_3 q_2}{1 - S_3 q_2} \\ r_2 &= \frac{q_3 - S_3 q_2}{1 - S_3 q_2}. \end{aligned}$$

Since a_2, a_3 and f are collinear, q_1, r_2 and r_3 satisfy the Triple spread formula. Set

$$s(q_1, q_2, q_3) \equiv (q_1 + q_2 + q_3)^2 - 2\left(q_1^2 + q_2^2 + q_3^2\right) - 4 q_1 q_2 q_3.$$

Use the Projective triple quad theorem and some algebraic manipulation to obtain

$$\begin{aligned} 0 &= s\left(q_1, \frac{q_3 - S_3 q_2}{1 - S_3 q_2}, \frac{q_2 - S_3 q_2}{1 - S_3 q_2}\right) \\ &= \frac{1}{(1 - S_3 q_2)^2}\left(\begin{array}{c} 2q_1q_2 + 2q_1q_3 + 2q_2q_3 - 4q_1q_2q_3 - q_1^2 - q_2^2 - q_3^2 \\ +2S_3q_1q_2q_3 + 2S_3q_1q_2^2 + 2S_3q_1^2q_2 - 4S_3q_1q_2 - S_3^2q_1^2q_2^2 \end{array}\right). \end{aligned}$$

This gives a quadratic equation in S_3, which can be rewritten as

$$(S_3 q_1 q_2 - q_1 - q_2 - q_3 + 2)^2 = 4(1 - q_1)(1 - q_2)(1 - q_3).$$

■

Theorem 8 (Dual projective cross law) *Suppose that* $\overline{a_1 a_2 a_3}$ *is a projective triangle with projective quadrances* q_1, q_2 *and* q_3*, and projective spreads* S_1, S_2 *and* S_3*. Then*

$$(S_1 S_2 q_3 - S_1 - S_2 - S_3 + 2)^2 = 4(1 - S_1)(1 - S_2)(1 - S_3).$$

Proof. This is dual to the previous theorem. ■

There is one more important notion which should be mentioned—the analog in this theory of area. The **projective quadrea** $\mathcal{A}$ of the projective triangle with projective quadrances q_1, q_2 and q_3, and projective spreads S_1, S_2 and S_3 is

$$\mathcal{A} = S_1 q_2 q_3 = S_2 q_1 q_3 = S_3 q_1 q_2.$$

These numbers are indeed equal due to the Projective spread law. The projective quadrea $\mathcal{A}$ is determined by the quadrances via the Projective cross law, and an explicit formula may also be derived.

Theorem 9 *The projective quadrea* $\mathcal{A} = \mathcal{A}(a_1, a_2, a_3)$ *of the projective points* $a_1 = [x_1 : y_1 : z_1]$, $a_2 = [x_2 : y_2 : z_2]$ *and* $a_3 = [x_3 : y_3 : z_3]$ *is*

$$\mathcal{A} = \frac{(x_1 y_2 z_3 - x_1 y_3 z_2 - x_2 y_1 z_3 + x_2 z_1 y_3 + y_1 x_3 z_2 - x_3 y_2 z_1)^2}{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)(x_3^2 + y_3^2 + z_3^2)}.$$

Proof. A calculation. ■

Since the numerator has factor the square of the determinant of the matrix with rows $[x_1, y_1, z_1]$, $[x_2, y_2, z_2]$ and $[x_3, y_3, z_3]$, $\mathcal{A} = 0$ precisely when the projective points a_1, a_2 and a_3 are collinear. So in general $\mathcal{A}$ can be viewed as a measure of the non-collinearity of the three projective points.

Solving right projective triangles

Theorem 10 (Right projective triangle) *Suppose a right projective triangle has projective quadrances q_1, q_2 and q_3, and projective spreads S_1, S_2 and $S_3 = 1$. Then any two of the five quantities $\{q_1, q_2, q_3, S_1, S_2\}$ determine the other three, solely through the three* ***basic equations***

$$q_3 = q_1 + q_2 - q_1 q_2 \qquad S_1 = q_1/q_3 \qquad S_2 = q_2/q_3.$$

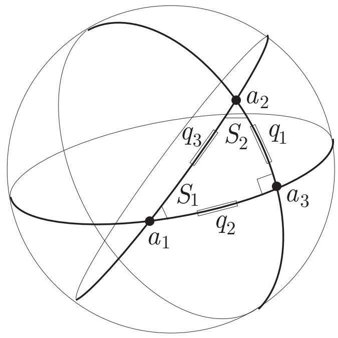

Figure 15: Right projective triangle

Proof. If you know any two of the projective quadrances then the Projective Pythagoras' theorem $q_3 = q_1 + q_2 - q_1 q_2$ allows you to determine the third, and the other two basic equations give the projective spreads.

If you know the two projective spreads S_1 and S_2 then use Projective Pythagoras' theorem and Projective Thales relations $q_1 = S_1 q_3$ and $q_2 = S_2 q_3$ to obtain

$$1 = S_1 + S_2 - S_1 S_2 q_3.$$

Thus

$$q_3 = (S_1 + S_2 - 1)/S_1 S_2 \qquad q_1 = S_1 q_3 = (S_1 + S_2 - 1)/S_2 \qquad q_2 = S_2 q_3 = (S_1 + S_2 - 1)/S_1.$$

If you know a projective spread, say S_1, and a projective quadrance then there are three possibilities. If the projective quadrance is q_3, then $q_1 = S_1 q_3$ and

$$q_2 = (q_3 - q_1)/(1 - q_1) = (q_3 - S_1 q_3)/(1 - S_1 q_3) \qquad S_2 = q_2/q_3 = (1 - S_1)/(1 - S_1 q_3).$$

If the projective quadrance is q_1, then $q_3 = q_1/S_1$ and

$$q_2 = (q_3 - q_1)/(1 - q_1) = q_1 (1 - S_1)/(S_1 (1 - q_1)) \qquad S_2 = q_2/q_3 = (1 - S_1)/(1 - q_1).$$

If the projective quadrance is q_2, then substitute $q_1 = S_1 q_3$ into the Pythagorean equation to get

$$q_3 = S_1 q_3 + q_2 - S_1 q_2 q_3.$$

So

$$q_3 = q_2/(1 - S_1 (1 - q_2)) \qquad q_1 = S_1 q_2/(1 - S_1 (1 - q_2)) \qquad S_2 = q_2/q_3 = 1 - S_1 (1 - q_2).$$

■

Remark 11 *The various equations derived in this proof are rational analogs of Napier's rules and are fundamental for projective trigonometry. However, they need not be memorized—just remember that they all follow from the three basic equations by elementary algebraic manipulations.*

Isosceles projective triangles.

A projective triangle is **isosceles** precisely when at least two of its projective quadrances are equal. From the Projective spread law this is equivalent to two projective spreads being equal.

Theorem 12 (Projective isosceles triangle) *Suppose a projective isosceles triangle has projective quadrances* $q_1 = q_2 = q$ *and* q_3, *and projective spreads* $S_1 = S_2 = S$ *and* S_3. *Then*

$$q_3 = 4q\,(1-S)\,(1-q)\,/\,(1-Sq)^2 \qquad \textit{and} \qquad S_3 = 4S\,(1-S)\,(1-q)\,/\,(1-Sq)^2\,.$$

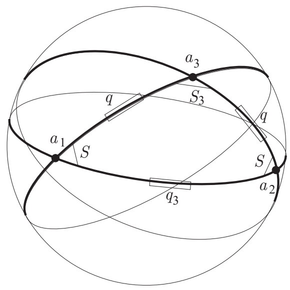

Figure 16: Projective isosceles triangle

Proof. Use the Projective spread law in the form

$$S_3 = \frac{Sq_3}{q}$$

to replace S_3 in the Projective cross law

$$\left(q^2S_3 - (2q + q_3 - 2)\right)^2 = 4\,(1-q)^2\,(1-q_3)\,.$$

This yields a quadratic equation in q_3 with solutions $q_3 = 0$, which is impossible, and

$$q_3 = \frac{4q\,(1-S)\,(1-q)}{(1-Sq)^2}.$$

Thus

$$S_3 = \frac{Sq_3}{q} = \frac{4S\,(1-S)\,(1-q)}{(1-Sq)^2}.$$

■

Equilateral projective triangles

A projective triangle is **equilateral** precisely when all its quadrances are equal. The following formula appeared as Exercise 24.1 in [Wildberger].

Theorem 13 (Equilateral projective triangles) *Suppose that a projective triangle is equilateral with common projective quadrance* $q_1 = q_2 = q_3 = q$*, and with common projective spread* $S_1 = S_2 = S_3 = S$*. Then*

$$(1 - Sq)^2 = 4(1 - S)(1 - q). \tag{3}$$

Proof. From the Projective isosceles triangle theorem

$$q = \frac{4q(1 - S)(1 - q)}{(1 - Sq)^2}.$$

Since $q \neq 0$ this yields

$$(1 - Sq)^2 = 4(1 - S)(1 - q).$$

■

Note that the result is symmetric in S and q. This relation between q and S is illustrated in Figure 17, which shows all the possible values for S and q between 0 and 1. Note that when q is small the value of S is approximately $3/4$, which is the planar case. As q increases S also increases, somewhat slowly, until when $q = 3/4$ the value of $S = 8/9$ is augmented by another value $S = 0$, which represents a degenerate projective triangle with all projective poitns collinear.

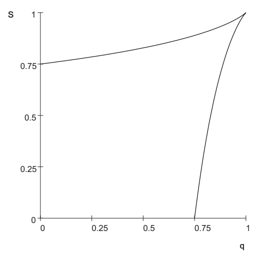

Figure 17: $(1 - Sq)^2 = 4(1 - S)(1 - q)$

Spherical regular polygons

Let's apply projective trigonometry to analyze regular polygons on the sphere. The **spherical quadrance** between two points on the sphere is just the projective quadrance of the two corresponding projective points, except that occasionally we refer to such a projective quadrance being **acute** or **obtuse**, depending on whether the central *rays* formed by those points make an acute or obtuse spread. This is a feature of spherical trigonometry in the decimal number field which distinguishes it from projective trigonometry.

From the north pole N of the sphere draw three equally spaced great circles, making spreads of $3/4$ with each other. Suppose points A, B and C lie on these lines and also on a meridian circle, and form an equilateral spherical triangle with center N. An exception occurs when A, B and C lie on the equator and so are collinear—we will however include this degenerate case also.

The Equilateral triangle theorem gives the relation between the spherical quadrances q and the spherical spreads S of $\overline{ABC}$. The spherical line NC also meets AB at M, where the lines are perpendicular. The right spherical triangle $\overline{NAM}$ has spreads of $3/4$ at N, 1 at M and say r at A.

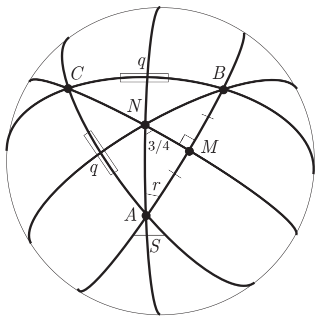

Figure 18: Spherical equilateral triangle

Now use the Right projective triangle theorem to deduce that

$$q(A,N) = (4r-1)/(3r) \qquad q(N,M) = (4r-1)/3 \qquad q(A,M) = (4r-1)/(4r).$$

The projective quadrance q between A and B is

$$q = 4\left(\frac{4r-1}{4r}\right)\left(\frac{1}{4r}\right) = \frac{4r-1}{4r^2}$$

and the projective spread S of $\overline{ABC}$ is

$$S = 4r(1-r).$$

This determines a parametrization of the algebraic curve (3), namely

$$[S,q] = \left[4r(1-r), \frac{4r-1}{4r^2}\right].$$

If each of the spherical triangles $\overline{ABN}$, $\overline{BCN}$ and $\overline{ACN}$ are equilateral, then $r = 3/4$ so that $S = 3/4$ and $q = 8/9$ and

$$q(A,N) = 8/9 \qquad q(N,M) = 2/3 \qquad q(A,M) = 2/3.$$

The condition that $q(A,N) = 8/9$ gives us two possible positions for A, one above the equator, and one below, corresponding to acute and obtuse spreads. However in both cases, the spread between A and B (which is necessarily also $8/9$) is obtuse. Thus to get a *regular tetrahedron* $\overline{NABC}$ choose A, B and C below the equator so that $q(A,N) = 8/9$ is obtuse. The corresponding angle is approximately 109.47°, a number familiar to chemists.

Regular quadrilaterals

From the north pole N draw two perpendicular great circles and suppose points A, B, C and D lie on these lines and also on a meridian circle, so that $\overline{ABCD}$ is a spherical regular quadrilateral with center N. Let q and S be the common quadrance and spread of this spherical regular quadrilateral respectively, so that for example $q = q(A, B)$ and $S = S(AB, AC)$. Let M be the midpoint of the arc AB so that $\overline{ANM}$ is a right spherical triangle with spreads of $1/2$ at N, 1 at M and say r at A.

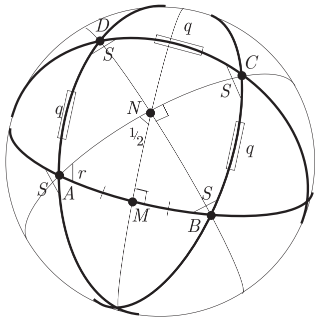

Figure 19: Spherical regular quadrilaterals

Now use the Right projective triangle theorem to deduce that

$$q(A, N) = (2r - 1)/r \qquad q(N, M) = 2r - 1 \qquad q(A, M) = (2r - 1)/(2r).$$

The projective quadrance q between A and B is

$$q = 4\left(\frac{2r - 1}{2r}\right)\left(\frac{1}{2r}\right) = \frac{2r - 1}{r^2}$$

and the projective spread $S = S(AB, AC)$ is

$$S = 4r(1 - r).$$

You can verify that q and S satisfy the relation

$$S^2q^2 = 16(1 - S)(1 - q)$$

shown in Figure 20.

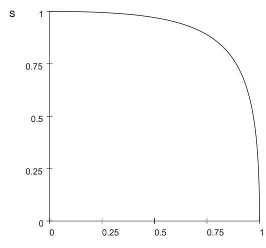

Figure 20: $S^2q^2 = 16(1 - S)(1 - q)$

If $r = 3/4$ then $S = 4r(1-r) = 3/4$ also, and

$$q(A,N) = 2/3 \qquad q(N,M) = 1/2 \qquad q(A,M) = 1/3$$

so that by the Equal spreads theorem

$$q(A,B) = 4(1/3)(1-1/3) = 8/9.$$

That means that if on these same great circles we draw antipodal points A', B', C' and D' then

$$q(A,A') = q(A,C) = 4(2/3)(1-2/3) = 8/9.$$

The points A, B, C, D, A', B', C and D' form a *cube*.

This dual of the cube is the *octahedron,* and it is the simplest Platonic solid to construct. Its points are the north and south poles N and S, and two pairs of perpendicular equatorial antipodal spherical points.

Regular pentagons

From the north pole N draw five equally spaced great circles, making spreads of $\beta = \left(5+\sqrt{5}\right)/8$ with each other. Suppose points A, B, C, D and E lie on these lines and also on a meridian circle, so that $\overline{ABCDE}$ is a **regular spherical pentagon** with center N. Let M be the midpoint of the arc AB so that $\overline{ANM}$ is a right spherical triangle with spreads of $\alpha = \left(5-\sqrt{5}\right)/8$ at N, 1 at M and say r at A.

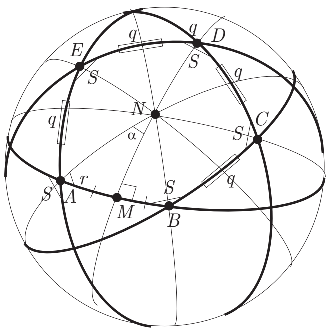

Figure 21: Regular spherical pentagon

Now use the Right projective triangle theorem to deduce that

$$q(A,N) = \frac{r+\alpha-1}{r\times\alpha} = \frac{1}{40r}\left(10+2\sqrt{5}\right)\left(8r-3-\sqrt{5}\right) \quad (4)$$

$$q(N,M) = \frac{r+\alpha-1}{\alpha} = \frac{1}{20}\left(5+\sqrt{5}\right)\left(8r-3-\sqrt{5}\right)$$

$$q(A,M) = \frac{r+\alpha-1}{r} = \frac{1}{8r}\left(8r-3-\sqrt{5}\right). \quad (5)$$

The projective quadrance q between A and B is

$$q = \frac{\left(3+\sqrt{5}\right)\left(8r-3-\sqrt{5}\right)}{16r^2}$$

and the projective spread S of $\overline{ABC}$ is

$$S = 4r\left(1-r\right).$$

You may verify that q and S satisfy the relation

$$\left(\frac{1+\sqrt{5}}{2}+qS\right)^2 = 2\left(3\sqrt{5}+7\right)\left(1-S\right)\left(1-q\right)$$

shown in Figure 22.

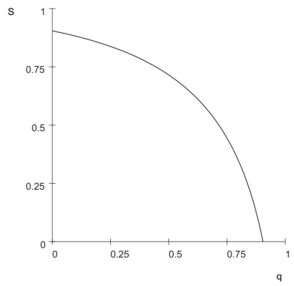

Figure 22: $\left(\left(1+\sqrt{5}\right)/2+qS\right)^2 = 2\left(3\sqrt{5}+7\right)\left(1-S\right)\left(1-q\right)$

In the special case when $r = \beta = \left(5+\sqrt{5}\right)/8$ (4) reduce to

$$q\left(A,N\right) = 4/5 \qquad q\left(N,M\right) = \left(5+\sqrt{5}\right)/10 \qquad q\left(A,M\right) = \left(5-\sqrt{5}\right)/10.$$

Now reflect N in AB to obtain the point D', lying on NM, and similarly also the points A', B', C' and E', all lying on the same meridian circle. The twelve points N, A, B, C, D, E, A', B', C', D',E' and S form twenty equilateral triangles, as indicated in Figure 23. We claim that all the triangles in the figure are congruent (meaning their quadrances have the same numerical values and types) and hence all are equilateral. Since D' is the reflection of N in AB, the spherical triangle $\overline{AMD'}$ is congruent to $\overline{AMN}$, so that $S\left(AM, AD'\right) = \beta$ and $q\left(A,D'\right) = q\left(A,N\right) = 4/5$. Thus all the spreads of the five triangles meeting at A are equal to β, so that $q\left(C',D'\right) = 4/5$. Use the Equal spreads theorem and some pleasant simplification to find

$$\begin{aligned} q\left(N,D'\right) &= 4q\left(N,M\right)\left(1-q\left(N,M\right)\right) \\ &= 4/5. \end{aligned}$$

That shows that D' is actually antipodal to D, and that $q\left(D',S\right) = 4/5$, so that by symmetry all triangles have equal quadrances. This establishes the existence of the *icosahedron* as a tesselation of the sphere, as in Figure 23.

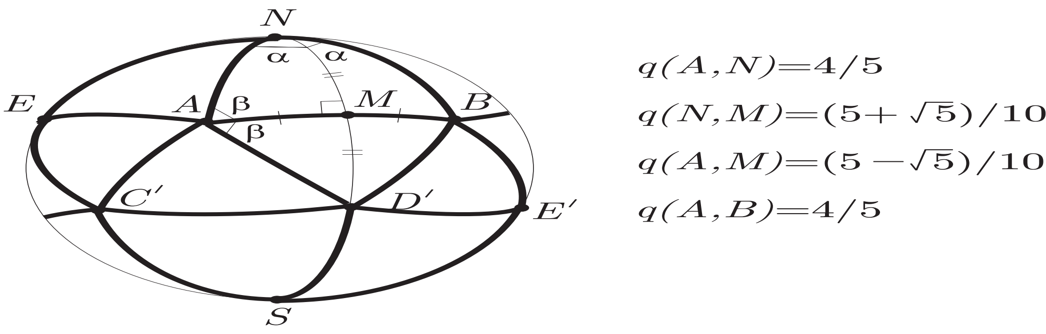

Figure 23: Icosahedron

To get the dual dodecahedron, consider the special case $r = 3/4$, for which (4) reduces to

$$q(A, N) = \left(10 - 2\sqrt{5}\right)/15 \qquad q(N, M) = \left(5 - \sqrt{5}\right)/10 \qquad q(A, M) = \left(3 - \sqrt{5}\right)/6.$$

So by the Equal spreads theorem

$$q(A, B) = 4 \times \frac{\left(3 - \sqrt{5}\right)}{6} \times \frac{\left(3 + \sqrt{5}\right)}{6} = \frac{4}{9}.$$

Now reflect N in the side AB to obtain the point N'. Then by the Equal spreads theorem

$$q(N, N') = 4 \times \frac{\left(5 - \sqrt{5}\right)}{10} \times \frac{\left(5 + \sqrt{5}\right)}{10} = \frac{4}{5}.$$

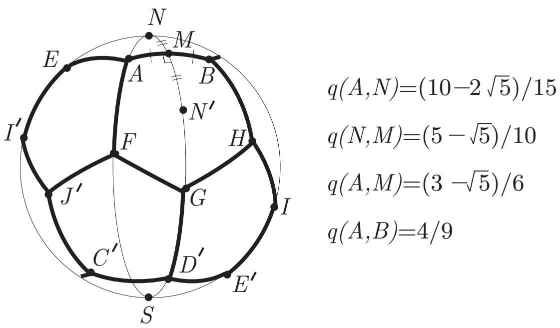

Figure 24: Dodecahedron

Reflect A in BC to obtain the point H, and reflect B in AE to obtain the point F. Then $q(A, F)$, $q(B, H)$ and $q(A, B)$ are all equal and acute. Since the spread $S = 3/4$ has the property that $S_2(S) = 4S(1 - S) = S$, the arcs AF, AB and BH are three sides of a regular spherical pentagon $\overline{FABHG}$ with center N', congruent to the original pentagon $\overline{ABCDE}$. Also N' is seen to be the intersection of AE and BC. The point G so constructed lies on NN'. Furthermore F lies on NA, and H lies on NB. In a similar way we construct five congruent regular pentagons forming a ring around the initial pentagon, each with quadrance $4/9$.

Now a second ring of five pentagons is constructed. The point H may be reflected in the line FG to obtain the point D', and reflected in KL to obtain E'. Then again $\overline{D'GHKE'}$ is a regular pentagon congruent to the original, and D' lies on NG, and E' lies on NK.

Finally it must be shown that the final 'bottom' pentagon $\overline{A'B'C'D'E'}$ is also congruent to the others. But this is clear since each of its sides has an acute quadrance of $4/9$, and each of its spreads is equal to $3/4$. This construction should be compared to the one in [Dorrie].

A second construction of the Platonic solids

Here is another more projective approach to the Platonic solids, which uses a projective analog of the example studied earlier.

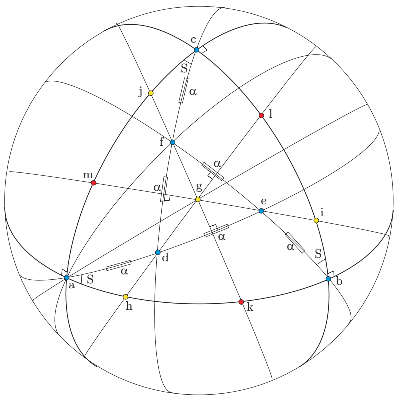

Figure 25: An important configuration

Suppose that $\overline{abc}$ is an equilateral projective triangle with projective quadrances and projective spreads all equal to one. These three projective points correspond to three mutually perpendicular coordinate axes. Now we will suppose that we can construct three new projective lines ad, be and cf as shown in Figure 25, making equal projective spreads S with ab, bc and ca respectively, and with the property that

$$Q(a,d) = Q(d,e) = Q(b,e) = Q(e,f) = Q(c,f) = Q(f,d) = \alpha. \qquad (6)$$

The six projective points a, b, c, d, e and f are marked in *blue* in the Figure. The other points are constructed by considering the altitudes in the projective triangle $\overline{def}$ which intersect in a yellow projective point marked g, and then intersecting the projective lines to give also *yellow* projective points h, i and j and *red* projective points k, l and m. The colours are chosen to match with the construction system ZOME.

Consider the projective triangle $\overline{acd}$ with projective quadrances $1, \alpha$ and $4\alpha(1-\alpha)$ (since both $q(c,f)$ and $q(f,d)$ are equal to α) and projective spread S at c. This is a dual right triangle, since $q(a,c) = 1$, and the projective spread at a must be $1 - S$. If the projective spread at d is R then the dual to Projective Thales theorem asserts that

$$\alpha = S/R \qquad \text{and} \qquad 4\alpha\,(1-\alpha) = (1-S)\,/R.$$

Thus

$$4\,(1-\alpha) = \frac{1-S}{S}$$

and so

$$\alpha = (5S-1)\,/4S \qquad \text{and} \qquad R = \left(4S^2\right)/\,(5S-1)$$

The dual to the projective Pythagoras' theorem gives

$$R = S + (1-S) - S\,(1-S) = 1 - S + S^2.$$

If you equate the two expressions for R you find that either $S = 1$, which is impossible by construction, or

$$5S^2 - 5S + 1 = 0. \tag{7}$$

This implies that

$$R = 4/5 \qquad \text{and} \qquad \alpha = 5S/4.$$

By continuing in a similar manner, and using the quadratic relation (7) repeatedly, you may obtain Figure 26.

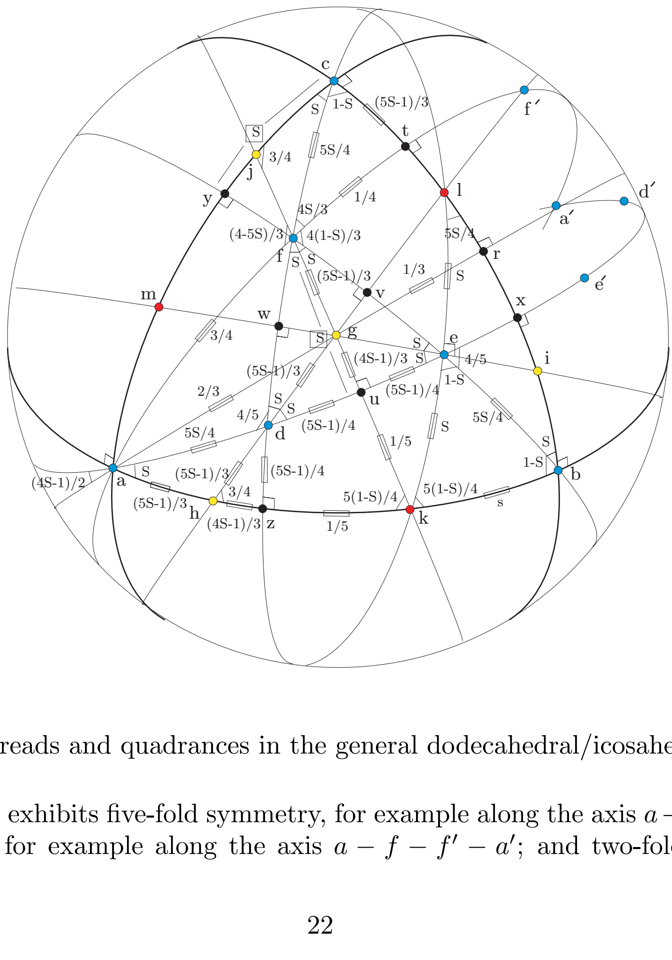

Figure 26: Spreads and quadrances in the general dodecahedral/icosahedral model

This configuration exhibits five-fold symmetry, for example along the axis $a-d-e-e'-d'-a'$; three-fold symmetry, for example along the axis $a-f-f'-a'$; and two-fold symmetry, along the axis $a - c - a'$. Some of the quadrances are constants independent of S, and thus of the field. These numbers include $1 = q(a, b)$, $3/4 = q(a, f)$, $4/5 = q(k, l)$, $2/3 = q(a, g)$ and $8/9 = q(h, i)$. Chemists know this latter 'angle' as $180 - \left(180 \arcsin \sqrt{8/9}\right)/\pi \approx 109.471\,220\,634$, formed by the atoms in a methane molecule.

Define a **star** to be a finite set of projective points. Now define the **blue star** B to be the finite set of projective points $\{a, b, c, d, e, f\}$ together with the reflections of d, e and f in the coordinate planes, that is all other projective points obtained by reflecting them in the coordinate planes, that is in the projective sides of the projective triangle $\overline{abc}$. Then B contains in general $3 + 4 \times 3 = 15$ projective points. Define the **yellow star** Y to be the set of yellow projective points $\{g, h, i, j\}$ together with their reflections in the coordinate axes, and define the **red star** R to be the set $\{k, l, m\}$ together with their reflections. Then Y has in general $4 + 2 \times 3 = 10$ projective points and R has $2 \times 3 = 6$ projective points.

The yellow star gives the projective vertices of a dodecahedron, the red star gives the projective vertices of an icosahedron, and the blue star gives the projective bisectors (midpoints of edges) of both. This latter object is in some sense the more fundamental object, as reflections/rotations in its elements generate the group of isometries, and the yellow and red stars are orbits in the projective plane of this isometry group. The projective versions of the tetrahedron, cube and octahedron are also found amongst these stars, the first two in the yellow star and the last in the blue star.

A construction using coordinates

To explicitly construct the dodecahedral/icosahedral model using coordinates, it suffices to have 5 a square in the field. Unlike the Euclidean plane, in the projective plane it is not necessary to have 3 a square to have three fold symmetry, as five fold symmetry yields three fold symmetry automatically! Set

$$a = [1 : 0 : 0] \qquad b = [0 : 1 : 0] \qquad c = [0 : 0 : 1]$$

and

$$d = \left[1 : \sigma : \sigma^2\right] \qquad e = \left[\sigma^2 : 1 : \sigma\right] \qquad f = \left[\sigma : \sigma^2 : 1\right].$$

The requirement (6) turns out to be equivalent to the condition

$$\sigma^2 + \sigma - 1 = 0. \tag{8}$$

Some straightforward calculations using the cyclic symmetry show that

$$g = [1 : 1 : 1] \qquad h = [\sigma + 1 : \sigma : 0] \qquad i = [0 : \sigma + 1 : \sigma] \qquad j = [\sigma : 0 : \sigma + 1]$$

and

$$k = [1 : 1 + \sigma : 0] \qquad l = [0 : 1 : 1 + \sigma] \qquad m = [1 + \sigma : 0 : 1].$$

Another computation shows then that

$$S = \frac{2 - \sigma}{5}.$$

Over the decimal numbers (8) has the solutions $x = \left(-1 + \sqrt{5}\right)/2$ and $S = \left(5 - \sqrt{5}\right)/10$. In this case $a = 5S/4 = \left(5 - \sqrt{5}\right)/8$ which demonstrates the connection with the regular pentagon.

Over the field $\mathbb{F}_{29}$ (8) has the solutions 5 and 23. With $x = 5$ the value of S becomes 11, and the picture becomes the following, where we have relabelled the projective points and lines to be in standard form, ending in 1 if possible.

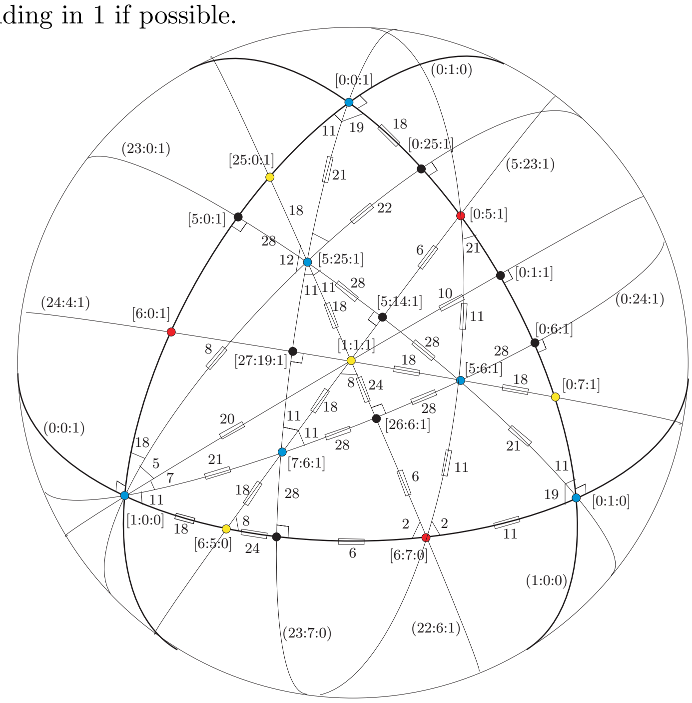

Figure 27: Dodecahedron and icosahedron in $\mathbb{F}_{29}$

There is an interesting question suggested by these constructions that applies to a general projective space. Define a star to be **regular** precisely when it is sent to itself by reflection/rotation in any of its members. Can one classify regular stars?

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[Dorrie] H. Dorrie, *100 Great Problems of Elementary Mathematics: Their history and solution*, translated by D. Antin, Dover, New York, 1965.

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