

Survivor: the Trigonometry Challenge

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July 28, 2006

Abstract

A mathematical drama involves marooned students, a triangle, and a tense competition between classical and rational trigonometry.

Three guys make a triangle

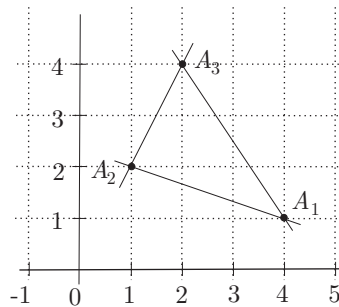
We've been marooned on a desert island for a month with plenty of food, pens and paper, but little entertainment. As a diversion from lying in the sun and eating tinned sardines, the red haired guy suggests we explain *trigonometry* to him, which he never understood in high school, despite being a fan of geometry. The tall skinny guy and I have just finished the standard first year university mathematics courses, so we agree this is a splendid idea to pass some time.

Unfortunately we have no trig tables, calculators or computers with us, but we should be able to explain most everything with just pen and paper. After all, if you *really* grasp a mathematical subject, simple examples should be accessible by hand. Especially if it's as basic as high school trigonometry.

So I get out a big piece of graph paper, randomly pick the three points

$$A_1 = [4, 1] \quad A_2 = [1, 2] \quad A_3 = [2, 4]$$

and explain that trigonometry is the study of the *measurement of triangles*. I propose to demonstrate the theory just for this particular triangle, to keep everything simple.



The separation of two points is measured by a *distance*, as we all know. The distance between any two points can be calculated by using Pythagoras' theorem, so that for example $d_1 = |A_2, A_3| = \sqrt{1^2 + 2^2} = \sqrt{5}$. I happen to know how to compute square roots without a calculator (call me a freak if you will), so with some work we were able to compute the three distances

$$\begin{aligned}d_1 &= |A_2, A_3| = \sqrt{5} \approx 2.236\,067\,977\dots \\d_2 &= |A_1, A_3| = \sqrt{13} \approx 3.605\,551\,275\dots \\d_3 &= |A_1, A_2| = \sqrt{10} \approx 3.162\,277\,660\dots\end{aligned}$$

The separation of two lines is measured by an *angle*, as we all know. To calculate the angle $\theta_1 = \angle A_2 A_1 A_3$ at the point A_1 of the above triangle, we use the Cosine law. We'll explain this later; for now we are just going to use it to calculate the angles. Thus

$$d_1^2 = d_2^2 + d_3^2 - 2d_2d_3 \cos \theta_1.$$

After some wrestling with the square roots, we get the approximate value

$$\cos \theta_1 = \frac{13 + 10 - 5}{2\sqrt{10}\sqrt{13}} = \frac{9}{\sqrt{130}} \approx 0.789\,352\,217$$

Now we need to recover θ_1 from this. Clearly we must invert cosine, and fortunately I remember the formula for the coefficients of the power series from our calculus class, allowing us to write

$$\begin{aligned}\arccos x &= \frac{1}{2}\pi - x - \frac{1}{6}x^3 - \frac{3}{40}x^5 - \frac{5}{112}x^7 - \frac{35}{1152}x^9 - \frac{63}{2816}x^{11} - \frac{231}{13\,312}x^{13} \\ &\quad - \frac{143}{10\,240}x^{15} - \frac{6435}{557\,056}x^{17} - \frac{12\,155}{1245\,184}x^{19} - \frac{46\,189}{5505\,024}x^{21} + \dots\end{aligned}$$

where as usual

$$\pi \approx 3.141\,592\,653\dots$$

Now we substitute $x = 0.789\,352\,217$ into this power series up to degree 21. Perhaps you haven't done this kind of calculation recently, but let me tell you, it sharpens the arithmetical skills! We all pitch in, and many hours later obtain

$$\theta_1 = \arccos x \approx 0.661\,114\,479\dots$$

But a bit of analysis with power series error terms shows that we are correct probably only to three decimal places. If we had used terms up to degree 39, then we would be correct probably only to five decimal places. That's a bit depressing, isn't it?

So, without a calculator, and using the obvious approach, it looks like we are in for a hard time. Why don't they tell you in school how hard it is to come up with angles in real life? Even with some clever short cuts, it takes us a day of feverish calculations to find the three angles in radians up to 9 decimal places.

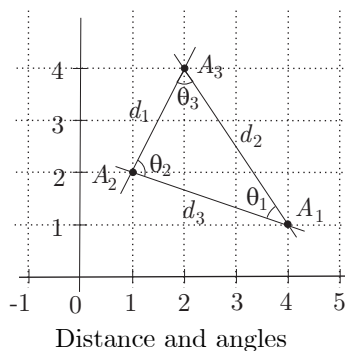
They are

$$\begin{aligned}\theta_1 &\approx 0.661\,043\,168\dots \\ \theta_2 &\approx 1.428\,899\,272\dots \\ \theta_3 &\approx 1.051\,650\,212\dots\end{aligned}$$

Note the relation

$$\theta_1 + \theta_2 + \theta_3 \approx 3.141\,592\,652\dots$$

which is close enough to π .



With the distances d_1 , d_2 and d_3 and the angles θ_1 , θ_2 and θ_3 finally in hand, we are ready to develop and illustrate the main laws of trigonometry! The Cosine law was already mentioned, and the Sine law is

$$\frac{\sin \theta_1}{d_1} = \frac{\sin \theta_2}{d_2} = \frac{\sin \theta_3}{d_3}.$$

Then we'll introduce the tangent, secant and cosecant functions, the Law of tangents, and all those lovely relations, special values and graphs for the trig functions and their inverse functions. Not to mention the interesting power series expansions used to compute them in calculators and computers.

I'm about to start explaining all this, when I notice the red haired fellow has that dull vacant look in his eyes. The tall skinny guy, who has been lost in thought, notices it too, and says 'You know, I think we should teach him *rational trigonometry* instead.'

Enter rational trigonometry

'What in Dickens is rational trigonometry?' I ask. Turns out he's been reading a recent book [Wildberger] which gives some newfangled 'simplified' form of trigonometry, in which the usual circular functions like $\sin \theta$ and $\tan \theta$ play no role, and the five main laws are supposedly quite simple. 'How can you possibly have trigonometry without trig functions?' the red haired guy asks.

Let me explain, says the tall fellow. According to Wildberger, triangles and circles are two separate subjects, and the first is more elementary than the

second. Rational trigonometry studies *triangles* without reference to *circles*, measures *quadrance* and *spread* instead of *distance* and *angle*, and ends up with simplified basic laws which are *polynomial*, not *transcendental*.

So instead of the three distances d_1, d_2 and d_3 , rational trigonometry works with the three quadrances Q_1, Q_2 and Q_3 , and instead of the three angles θ_1, θ_2 and θ_3 , it uses the three spreads s_1, s_2 and s_3 . These are indicated in a diagram as so, he says, and draws the following picture. The beauty of the theory is that

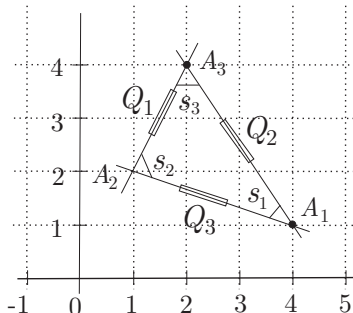


Figure 1: Quadrances and spreads

with these as the fundamental concepts, most everything becomes much easier to derive, remember and apply.

‘That sounds good’, says the red haired guy, ‘so tell us—what exactly are quadrances and spreads, and what are their values for our triangle?’

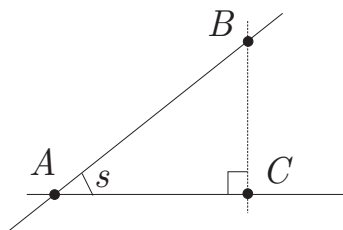
The separation of two points is measured by a **quadrance** and is just the square of the distance. So the quadrances of our triangle are

$$\begin{aligned} Q_1 &= Q(A_2, A_3) = 5 \\ Q_2 &= Q(A_1, A_3) = 13 \\ Q_3 &= Q(A_1, A_2) = 10. \end{aligned}$$

The separation of two lines is measured by a **spread**, and is always a number between 0 and 1, being 0 when the lines are parallel and 1 when the lines are perpendicular. An angle of 45° is a spread of $1/2$, and 30° and 60° are respectively spreads of $1/4$ and $3/4$.

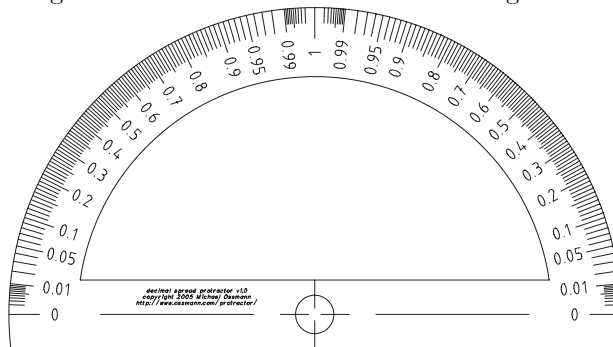
In general the spread between two lines is the quotient of two quadrances. Suppose the lines meet at a point A , and B is any other point on either of the two lines, with C the foot of the perpendicular from B to the other line. Then the spread s between the lines is

$$s = \frac{Q(B, C)}{Q(A, B)}.$$



Definition of spread

You can make a spread protractor that measures spreads in the same way an ordinary protractor measures angles. Here is one created by Mike Ossmann [Ossmann], although of course we didn't have such a thing at the time.



A spread protractor

'But the notion of a spread is not linear like an angle is!' I object. Sure, so what? says the skinny guy. Linear or non-linear is not the point. What's important is: how easy is it to get good answers? With quadrance and spread, the basic laws are polynomial, not transcendental, so they are much simpler—and more accurate.

'But what are you going to do when you get to Fourier analysis or harmonic motion in physics?', I demand. The skinny guy replied that most of the applications of trig functions to circles are reasonably elementary, and are better off approached via complex numbers and the exponential function. The fact is you don't really need to know stuff like $\cot(2\pi/3) = -\frac{1}{3}\sqrt{3}$ or $\sin(\arctan x) = x/\sqrt{1+x^2}$.

'Okay, so how do you find the spreads in our triangle?' I ask.

Use one of the basic laws, the one that plays the role of the Cosine law. It's called the **Cross law**, and says that

$$(Q_2 + Q_3 - Q_1)^2 = 4Q_2Q_3(1 - s_1).$$

So since we already have Q_1, Q_2 and Q_3 , we can calculate that

$$1 - s_1 = \frac{(10 + 13 - 5)^2}{4 \times 10 \times 13} = \frac{81}{130}$$

so

$$s_1 = 49/130.$$

Similarly

$$\begin{aligned} s_2 &= 49/50 \\ s_3 &= 49/65. \end{aligned}$$

That gives the three quadrances and the three spreads of the triangle.

‘So what are the other four laws?’ I ask. One is the **Spread law**, which replaces the Sine law and states that

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3}.$$

Another is the **Triple spread formula**, which states that the three spreads of the triangle satisfy

$$(s_1 + s_2 + s_3)^2 = 2(s_1^2 + s_2^2 + s_3^2) + 4s_1s_2s_3.$$

This is the analog to the fact that the sum of the angles is π . So we can check that for our triangle

$$\begin{aligned} \left(\frac{49}{130} + \frac{49}{50} + \frac{49}{65}\right)^2 &= \frac{470\,596}{105\,625} \\ &= 2\left(\left(\frac{49}{130}\right)^2 + \left(\frac{49}{50}\right)^2 + \left(\frac{49}{65}\right)^2\right) \\ &\quad + 4 \times \frac{49}{130} \times \frac{49}{50} \times \frac{49}{65}. \end{aligned}$$

‘But surely that’s a lot more complicated than the statement that the sum of the angles is π ,’ I say.

Only if you forget that π is essentially unknowable—you ever only work with some rational approximation to it, so that numerical statements involving it are inevitably only approximations to the truth. What we’ve just checked is completely 100% accurate, he replies.

‘And the other two laws?’ Actually they are just special cases of the Cross law. When $s_3 = 0$ the three points are collinear and then the three quadrances satisfy $(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2$ which can be written more symmetrically as

$$(Q_1 + Q_2 + Q_3)^2 = 2(Q_1^2 + Q_2^2 + Q_3^2).$$

That’s called the **Triple quad formula**. Note that the Triple quad formula and the Triple spread formula differ only by a single cubic term.

And when $s_3 = 1$ you get **Pythagoras’ theorem** in the form $Q_1 + Q_2 = Q_3$. Which, you have to admit, is a pleasant form for the most important theorem in all of mathematics.

These are the five main laws of rational trigonometry. According to Wildberger, that's pretty well all you need to solve the majority of trigonometry problems.

We're silent at first, but the red haired guy seems to have emerged from his funk. 'Heh, I think I could get this,' he says.

A contest emerges

I say 'Clearly rational trigonometry is just some kind of cute reformulation of what we already know'.

'Really?' replies the skinny guy. 'Except that it took me fifteen minutes to calculate the three quadrances and spreads, with no inaccuracy at all, while it took us a whole day to calculate the distances and angles, and we still only get everything to 9 decimal places. Assuming we found all our mistakes, of course'.

'Instead of engaging in rhetoric,' says the red haired fellow, 'how about applying some scientific methodology? We now have two competing theories. Let's see how each solves some specific problems. We'll have a *trigonometry challenge!* The theory which loses gets expelled from the curriculum. Sound fair?'

Of course we both agreed, and after some discussion we decided to allow only use of the triangle's distances and angles, or alternatively its quadrances and spreads, to solve the challenges. In other words, no analytic geometry involving the coordinates of the points A_1 , A_2 and A_3 was to be involved—just *trigonometry*. Then we were off!

Area

'What's the area of the triangle?' was the first question.

'Well, the ancient Greeks knew how to do that one,' I said merrily. Once you have the three side lengths d_1 , d_2 and d_3 , the classical Heron's formula for the area a is

$$a = \sqrt{s(s - d_1)(s - d_2)(s - d_3)}$$

where $s = (d_1 + d_2 + d_3) / 2$. So using our known values of d_1 , d_2 and d_3 , some calculation (including that painful square root procedure) showed that

$$a \approx 3.499999999.$$

The tall skinny guy explained that in rational trigonometry the area was not as fundamental as the **quadrea** \mathcal{A} of the triangle, which was defined as the difference between the left and right sides of the Triple quad formula, and turned out to be sixteen times the square of the usual area. Thus

$$\begin{aligned} \mathcal{A} &= (Q_1 + Q_2 + Q_3)^2 - 2(Q_1^2 + Q_2^2 + Q_3^2) \\ &= (5 + 13 + 10)^2 - 2(25 + 169 + 100) \\ &= 196. \end{aligned}$$

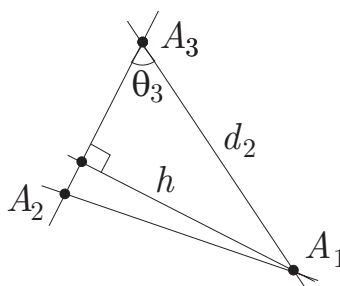
Of course that meant that the area was the square root of $196/16 = 49/4$ which was 3.5. Hmm... Was this some kind of black magic, or just pulling the wool over my eyes?

An altitude

The second challenge was: ‘What is the length, or quadrance, of the altitude from A_1 to the opposite side $\overline{A_2A_3}$?’

‘This is a completely straightforward question from a classical point of view, just geared to the definition of the sine of an angle,’ I explain. Denoting the length of the altitude by h , clearly

$$\begin{aligned} h &= d_2 \sin \theta_3 \\ &= 3.605\,551\,275 \times \sin(1.051\,650\,212). \end{aligned}$$



An altitude

With a calculator, this is a snap, but without one, we’ll probably have to use the power series expansion for $\sin \theta$, namely

$$\sin \theta = \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 - \frac{1}{5040}\theta^7 + \frac{1}{362\,880}\theta^9 - \frac{1}{39\,916\,800}\theta^{11} + \dots$$

But wait—there is a simpler way, I realize. Use the Cosine law to get

$$\begin{aligned} \cos \theta_3 &= \frac{d_1^2 + d_2^2 - d_3^2}{2d_1d_2} = \frac{(\sqrt{5})^2 + (\sqrt{13})^2 - (\sqrt{10})^2}{2\sqrt{5}\sqrt{13}} \\ &= 0.496\,138\,938 \end{aligned}$$

and then the relation $\cos^2 \theta_3 + \sin^2 \theta_3 = 1$ to compute that

$$\sin \theta_3 = 0.868\,243\,142.$$

Then a single multiplication gives

$$h = 3.130\,495\,167.$$

That took me only half an hour, since I’m pretty good with arithmetic.

I observe that your avoidance of the power series for $\sin \theta$ is in the spirit of rational trigonometry, says the tall skinny guy, as this avoids finding the angle θ_3 . Here is how we do this completely rationally. The quadrance H of the altitude is, from the definition

$$\begin{aligned} H &= Q_2 s_3 \\ &= 13 \times \frac{49}{65} = \frac{49}{5}. \end{aligned}$$

To get your distance h , just take the square root of that, namely

$$h = \sqrt{H} = \frac{7}{\sqrt{5}}.$$

That took him two minutes.

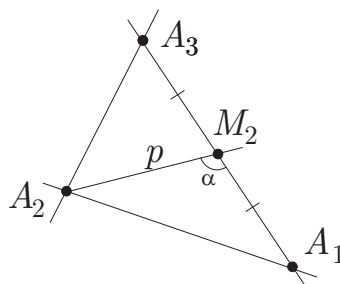
A computation then showed that my obtained value of h agreed with the decimal expansion of $7/\sqrt{5}$. That was, I have to admit, a tad disconcerting.

A median

The third challenge was: ‘Let’s consider the median from A_2 to the side $\overline{A_1 A_3}$. What’s its length, or quadrance? And what’s the angle, or spread, between it and the side $\overline{A_1 A_3}$?’

I knew what to do. Call the midpoint of $\overline{A_1 A_3}$ the point M_2 and suppose that $|A_2, M_2| = p$. Then the Cosine law in the triangle $\overline{A_1 A_2 M_2}$ gives immediately

$$p^2 = (d_2/2)^2 + d_3^2 - 2d_2 d_3 \cos \theta_1$$



A median

Since we know d_2, d_3 and θ_1 , it's just a calculation. Of course we'll use the fact that we calculated

$$\cos \theta_1 = \frac{9}{\sqrt{130}}$$

earlier, and the expressions for d_2 and d_3 as $\sqrt{13}$ and $\sqrt{10}$ respectively. The tall guy tried to object to this on the grounds that this was in reality going back

to the original coordinates, not sticking with the distances and angles we had tabulated. But I wasn't in the mood for technicalities, so I continued

$$\begin{aligned} p^2 &= \left(\frac{\sqrt{13}}{2}\right)^2 + (\sqrt{10})^2 - 2 \times \frac{\sqrt{13}}{2} \times \sqrt{10} \times \frac{9}{\sqrt{130}} \\ &= \frac{13}{4} + 10 - 9 = \frac{17}{4} \end{aligned}$$

and so $p = \sqrt{17/2}$. Easy. Now as for that angle $\angle A_2M_2A_1$, call it α . The Sine law in $\overline{A_1A_2M_1}$ shows that

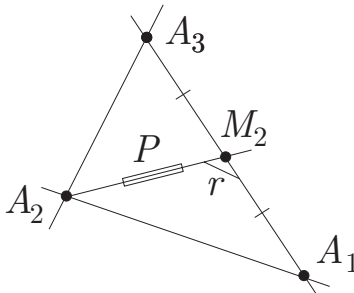
$$\begin{aligned} \sin \alpha &= \frac{d_3 \sin \theta_1}{p} \\ &= \frac{2\sqrt{10} \sin \theta_1}{\sqrt{17}} \end{aligned}$$

Thus

$$\alpha = \arcsin \left(\frac{2\sqrt{10} \sin \theta_1}{\sqrt{17}} \right).$$

Good grief, do we really have to compute this? No wonder the rather natural question about what angles the medians make with the opposite sides of a triangle is rarely studied in geometry courses. Still, there must be some clever way of exploiting the fact that we already know that $\cos \theta_1 = 9/\sqrt{130}$...

Yes, there is, says the tall guy. Its called rational trigonometry. Let's start again with the quadrance $P = Q(A_1, M_1)$, which is of course your p^2 , and let r be the spread between the median and the side $\overline{A_1A_3}$.



The median again

Then the Cross law in $\overline{A_1A_2M_2}$ gives

$$(P - Q_3 - Q_2/4)^2 = 4 \times Q_3 \times \frac{Q_2}{4} \times (1 - s_1)$$

or

$$\left(P - \frac{53}{4}\right)^2 = 81$$

so that $P = 17/4$ or $P = 89/4$. ‘Aha! So which one is it then?’ I ask. Well probably the *Triangle spread rules* that Wildberger mentions show which solution to take, but I don’t know them, so let’s see... if we also take the Cross law in $\overline{A_2A_3M_2}$ we get

$$(P - Q_1 - Q_2/4)^2 = 4 \times Q_1 \times \frac{Q_2}{4} \times (1 - s_3)$$

or

$$\left(P - \frac{33}{4}\right)^2 = 16$$

so that $P = 49/4$ or $17/4$. Putting the two together shows that $P = 17/4$. As for the spread r between A_2M_2 and A_1A_3 , the Spread law in $\overline{A_1A_2M_1}$ gives

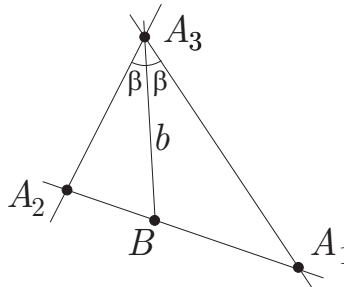
$$\frac{r}{10} = \frac{49/130}{17/4}$$

so that $r = 196/221$.

I have to admit, I was starting to get a bit worried. Could it be that we had spent years learning the wrong theory?

An angle bisector

‘For my last question,’ the red-haired guy says, ‘let’s consider the angle bisector at A_3 . What is its length, or quadrance, and what is the angle, or spread, which it makes with A_1A_2 ?’



An angle bisector

Well this is clearly a question involving angle chasing, so I am confident I’ll win handily. The equal angles formed by the bisector at A_3 are just $\beta = \theta_3/2 = 0.525\ 825\ 106$. If the bisector meets A_2A_3 at B , then the angle $\angle A_2BA_3$ is

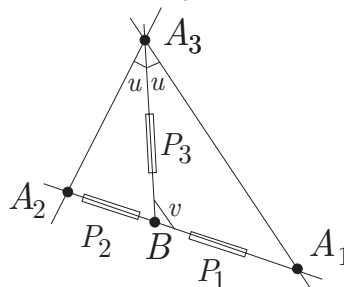
$$\pi - 1.428\ 899\ 272 - 0.525\ 825\ 106 = 1.186\ 868\ 276$$

and the length b of the bisector is by the Sine law

$$\begin{aligned}
 b &= \frac{d_1 \sin \theta_2}{\sin(1.18686827559)} \\
 &= \frac{2.236067977 \times \sin(1.428899272)}{\sin(1.186868276)} \\
 &= 2.387395616.
 \end{aligned}$$

Okay, I do admit that it took me more than four hours to make this calculation (you have to go up to degree 15 in one of the power series to get accuracy to 9 digits), and this gave the skinny guy time to figure out how to do things rationally. Here's what he did.

Set $Q(A_1, B) = P_1$, $Q(A_2, B) = P_2$ and $Q(A_3, B) = P_3$ and suppose the equal spreads made by the bisector at A_3 are u .



An angle bisector again

The Spread laws in the triangles $\overline{A_1A_3B}$ and $\overline{A_2A_3B}$ give

$$\begin{aligned}
 \frac{u}{P_1} &= \frac{s_1}{P_3} \\
 \frac{u}{P_2} &= \frac{s_2}{P_3}.
 \end{aligned}$$

Divide one of these equations by the other to get

$$\frac{P_2}{P_1} = \frac{s_1}{s_2} = \frac{50}{130} = \frac{5}{13}$$

But $\{P_1, P_2, P_3\}$ satisfy the Triple quad formula, since they are the quadrances formed by three collinear points, so that

$$(P_1 + P_2 - 10)^2 = 4P_1P_2.$$

Divide both sides by P_1^2 and use the previous equation to obtain

$$\left(1 + \frac{5}{13} - \frac{10}{P_1}\right)^2 = 4 \times \frac{5}{13}$$

which yields

$$16P_1^2 - 1170P_1 + 4225 = 0$$

with solutions

$$P_1 = \frac{585 \pm 65\sqrt{65}}{16}.$$

The two solutions correspond to the two angle bisectors at A_3 (one of them is external). The value we want is

$$P_1 = (585 - 65\sqrt{65})/16$$

so that

$$P_2 = 5(585 - 65\sqrt{65})/(16 \times 13).$$

The Cross laws in $\overline{A_1A_3B}$ and $\overline{A_1A_3B}$ are

$$\begin{aligned} (P_3 - P_1 - Q_2)^2 &= 4P_1Q_2(1 - s_1) \\ (P_3 - P_2 - Q_1)^2 &= 4P_2Q_1(1 - s_2) \end{aligned}$$

Take the difference of these equations to get the linear equation

$$P_3(61 - 5\sqrt{65}) - \frac{14625}{8} + \frac{1697}{8}\sqrt{65} = 0$$

with solution

$$P_3 = \frac{325}{16} - \frac{29}{16}\sqrt{65}.$$

To find the spread v between the A_3 bisector and the opposite side A_1A_2 , use the Spread law in $\overline{A_1A_3B}$

$$\frac{v}{13} = \frac{49/130}{P_3}.$$

Thus

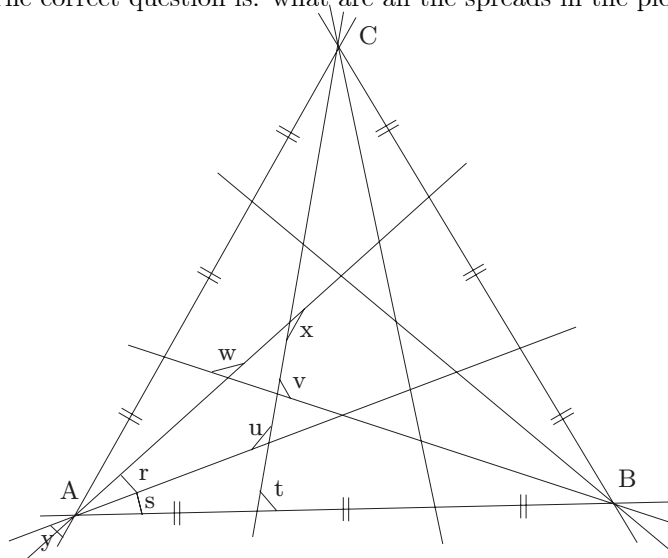
$$\begin{aligned} v &= \frac{49}{10} \left(\frac{325}{16} - \frac{29}{16}\sqrt{65} \right)^{-1} \\ &= \frac{1}{2} + \frac{29}{650}\sqrt{65}. \end{aligned}$$

Now it took him about two hours to go through all these calculations, but once we computed that my answers could be obtained as decimal approximations to his, the penny dropped, and I admitted defeat. It looked like rational trigonometry was the right way of thinking about these problems, giving answers that *really were correct*, and not just approximations to the truth.

A happy ending

Fortunately we were ultimately rescued from our desert island plight. But not before we had lots of time to verify that rational trigonometry gives better answers to a whole host of geometric problems. Here is just one of the many cool pictures we uncovered.

Take an equilateral triangle \overline{ABC} , trisect each of the sides, join all these new points to the original vertices and ask: what are all the angles in the picture? The question is wrong, so the answer is ugly, and no one bothers to analyse the situation. The correct question is: what are all the spreads in the picture?



Equilateral triangle into threes

Well, they turn out to be

$$s = 3/28 \quad r = 27/196 \quad t = 27/28 \quad u = 3/4 \\ v = 48/49 \quad w = 3/4 \quad x = 75/196. \quad y = 3/7.$$

What could be better than that? The red-haired guy cheerfully summed it up this way: ‘Classical trigonometry to rational trigonometry is like tinned sardines to roast dinners.’

References

- [Ossmann] M Ossmann, ‘Print a Protractor’, download at <http://www.ossmann.com/protractor/>
- [Wildberger] N J Wildberger, *Divine Proportions: Rational Trigonometry to Universal Geometry*, Wild Egg Books, Sydney, 2005, <http://wildegg.com>.